An hypergraph based formulation for an Automatic Storage Design problem

Luis Marques, François Clautiaux, Aurélien Froger

Univ. Bordeaux, CNRS, Inria, Bordeaux INP, IMB, UMR 5251, F-33400 Talence, France

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Introduction

Arc flow formulations are increasingly popular in the field of integer programming, [de Lima et al., 2022].

Such formulations are built from graphs, often transitions graphs of dynamic programs.

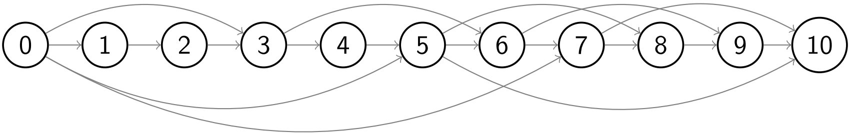

Figure: Example of arc flow formulation

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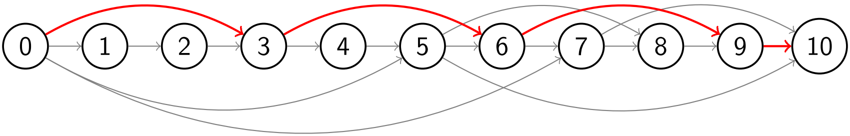

Figure: Example of arc flow formulation

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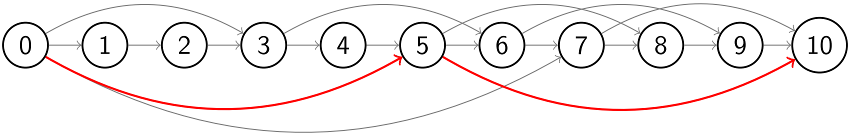


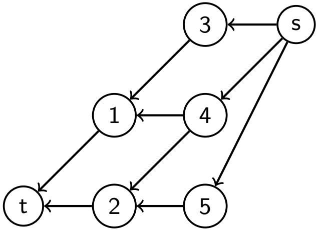
Figure: Example of arc flow formulation

Arc flow formulations and dynamic programming

Bacwkards recursive dynamic program:

- s: initial state
- t: terminal state
- f: function associating a state v to the minimum cost of going from state s to v
- $c_{u,v}$: cost of going from state u to state v
- $\Gamma^-(v)$: set of states preceding v

$$f(v) = \begin{cases} 0 & \text{if } v = s \\ \min_{u \in \Gamma^-(v)} \{c_{u,v} + f(u)\} & \text{otherwise} \end{cases}$$



Arc flow formulations and dynamic programming

- $G = (V, A)$: transition graph
- R: set of resources
- For every resource $r \in R$:
  - q_r: resource capacity
- For every arc $a \in A$:
  - c_a: arc cost
  - $b_{a,r}$: resource consumption of resource r

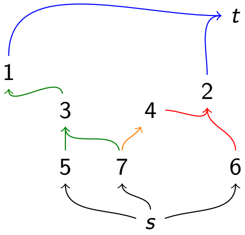
$$
\begin{aligned}
\text{minimize} \quad & \sum_{a \in A} c_a x_a && && (1) \\
\text{subject to} \quad & \sum_{a \in A^-(v)} x_a - \sum_{a \in A^+(v)} x_a = 0 && v \in V \setminus \{s, t\} && (2) \\
& \sum_{a \in A^-(t)} x_a = 1 && && (3) \\
& \sum_{a \in A(r)} b_{a,r} x_a \leq q_r && r \in R && (4) \\
& x_a \in \mathbb{N} && a \in A && (5)
\end{aligned}
$$

Extending arc flow formulation to hypergraphs

Bacwkards recursive dynamic program:

- f: function associating a state v to the minimum cost of going from state s to v
- $\Gamma^{-}(v)$: set of sets of states v can be decomposed into
- $c_{U,v}$: cost of decomposing state v into the set of states U

$$f(v) = \begin{cases} 0 & \text{if } v = s \\ \min_{U \in \Gamma^{-}(v)} \{c_{U,v} + \sum_{u \in U} f(u)\} & \text{otherwise} \end{cases}$$



Extending arc flow formulations to hypergraphs

Graph

- The flow conservation constraints form a totally unimodular matrix
- The domain of variables is $\{0, 1\}$

Hypergraph

- The flow conservation constraints form a totally dual integral matrix (TDI) [Martin et al., 1990]
- The domains of variables is unbounded and bounding those we can breaks the TDI property
- Flow may be multiplied without cost

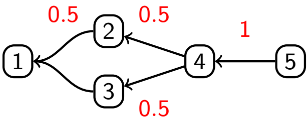


Figure: Example of half a unit of flow creating one unit of flow

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Automatic Storage Design
Introduction

We consider a two-phase and three-dimensional variant of the temporal knapsack problem called *Automatic Storage Design* problem:

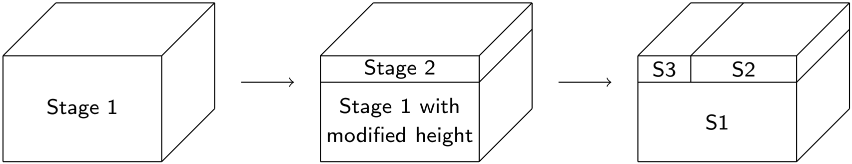
- M: Storage box with a *height* H, a *width* W and a *length* L.
- $\mathcal{I}$: Set of items. Each item i has a *height* h_i, a *width* w_i, a *length* l_i, a *profit* p_i, a *time period* s_i at which it enters the storage and a *duration* d_i during which the item is stored.
- $\mathcal{T}$: Set of consecutive time periods.

Automatic Storage Design
Introduction

The decisions represent a box design followed by an assignment of items.

- Partition the box to form shelves and partition each shelf to form compartments.
- Assign items to compartments.

Objective: Maximise the sum of profits of items assigned to the box.

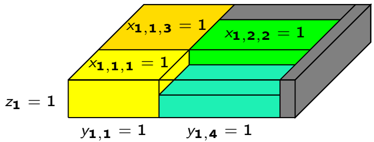


Automatic Storage Design

Compact formulation

Decision variables:

- z_i: 1 if a shelf of height h_i has been created, 0 otherwise.
- $y_{i,j}$: 1 if a compartment of width w_j has been created in the shelf of height h_i such that $i \leq j$, 0 otherwise.
- $x_{i,j,k}$: 1 if item k has been assigned to the compartment of height h_i and width w_j such that $i \leq j \leq k$, 0 otherwise.



Automatic Storage Design

Compact formulation

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in \mathcal{I}} \sum_{\substack{j \in \mathcal{I} \\ j \geq i}} \sum_{\substack{k \in \mathcal{I} \\ k \geq j}} p_i x_{i,j,k} \\
\text{subject to} \quad & \sum_{i \in \mathcal{I}} z_i h_i \leq H \\
& \sum_{\substack{j \in \mathcal{I} \\ j \geq i}} y_{i,j} w_j \leq W && i \in \mathcal{I} \\
& \sum_{\substack{k \in \mathcal{I} \\ k \geq j \\ s_k \leq t \leq s_k + d_k}} l_k x_{i,j,k} \leq L && i, j \in \mathcal{I}, j \geq i, t \in \mathcal{T} \\
& \sum_{\substack{i,j \in \mathcal{I} \\ i \leq j \leq k}} x_{i,j,k} \leq 1 && k \in \mathcal{I} \\
& x_{i,j,k} \leq y_{i,j} && i, j, k \in \mathcal{I}, k \geq j \geq i \\
& y_{i,j} \leq z_i && i, j \in \mathcal{I}, j \geq i \\
& z_i \in \{0, 1\} && i \in \mathcal{I} \\
& y_{i,j} \in \{0, 1\} && i, j \in \mathcal{I}, j \geq i \\
& x_{i,j,k} \in \{0, 1\} && i, j, k \in \mathcal{I}, k \geq j \geq i
\end{aligned}
$$

Automatic Storage Design

Dynamic program

Each state is noted by $(h, w, l)^s$, h being the height, w the width, l the length and s being the stage.

$$\alpha^1(h, w, l) = \max_{h' \in \mathcal{H}, h' \leq h} \{\alpha^2(h', w, l) + \alpha^1(h - h', w, l)\}$$

$$\alpha^2(h, w, l) = \max_{w' \in \mathcal{W}_h, w' \leq w} \{\alpha^3(h, w', l) + \alpha^2(h, w - w', l)\}$$

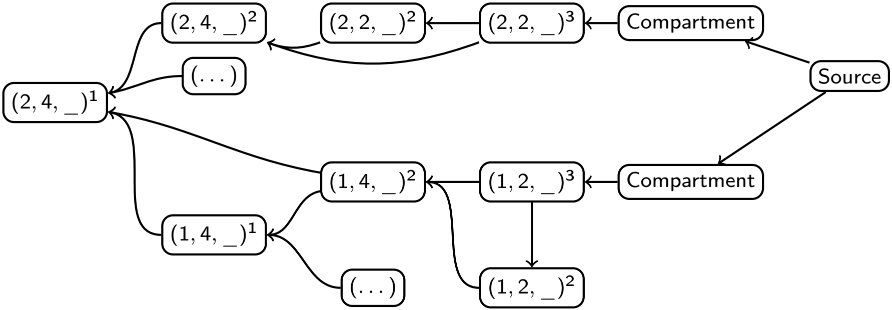
States $(h, w, l)^3$, which are temporal knapsack problems, are modeled by events as in [Clautiaux et al., 2021].

Automatic Storage Design

Dynamic program

$$\alpha^1(h,w,l) = \max_{h' \in \mathcal{H}, h' \leq h} \{\alpha^2(h',w,l) + \alpha^1(h-h',w,l)\}$$

$$\alpha^2(h,w,l) = \max_{w' \in \mathcal{W}_h, w' \leq w} \{\alpha^3(h,w',l) + \alpha^2(h,w-w',l)\}$$



Automatic Storage Design

Arc flow formulation

Let $G = (V, A)$ be the transition hypergraph associated with the dynamic program. Let $A(i)$ be the set of arcs corresponding to assigning item i in a compartment.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{a \in A} p_a x_a & \\
\text{subject to} \quad & \sum_{a \in A^-(v)} x_a - \sum_{a \in A^+(v)} x_a = 0 & v \in V \setminus \{s, t\} \\
& \sum_{a \in A^-(t)} x_a = 1 & \\
& \sum_{a \in A(i)} x_a \leq 1 & i \in \mathcal{I} \\
& x_a \in \mathbb{N} & a \in A
\end{aligned}
$$

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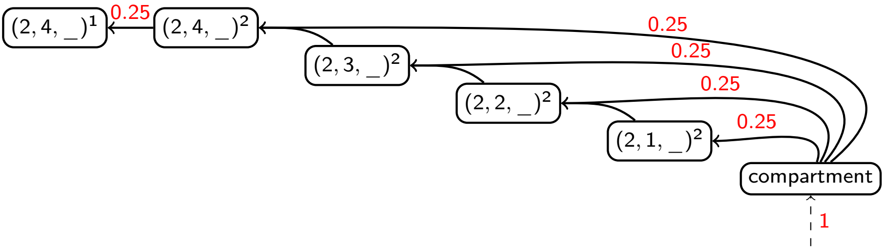
Improve the linear relaxation

Valid cuts and algorithm changes

Suppose a box with height 2 and width 4 and two items with heights 1 and 2 and widths 4 and 1.

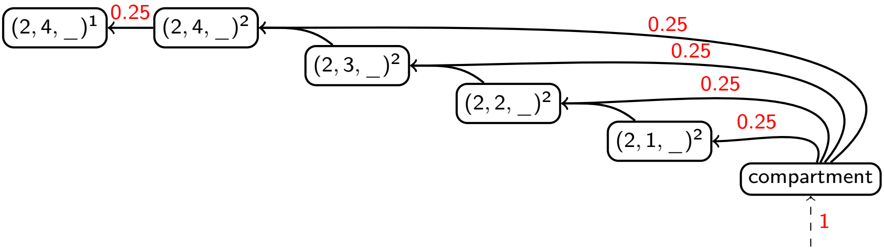


Flow recombines to form a compartment.

Improve the linear relaxation

Valid cuts and algorithm changes

Suppose a box with height 2 and width 4 and two items with heights 1 and 2 and widths 4 and 1.



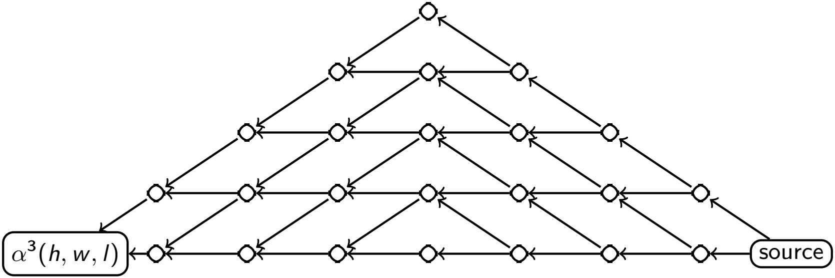
Flow recombines to form a compartment.

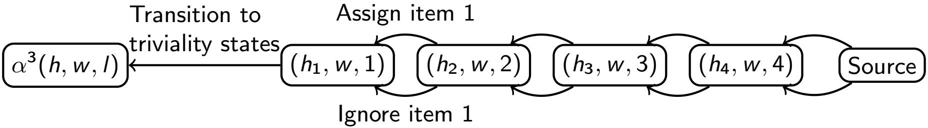
Valid cut:

$$\sum_{a \in A(i)} x_a - \sum_{h' \geq h_i} \sum_{a \in A(h)} x_a \leq 0 \quad i \in \mathcal{I}$$

Detect trivial subproblems

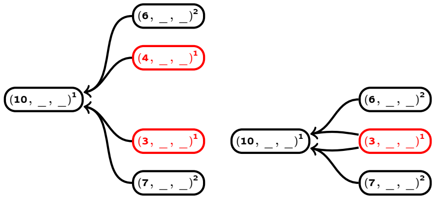
A subproblem is trivial if there exists a solution such that every item that fits is assigned.

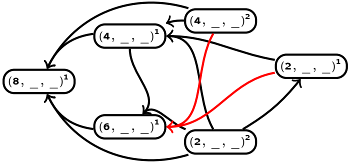


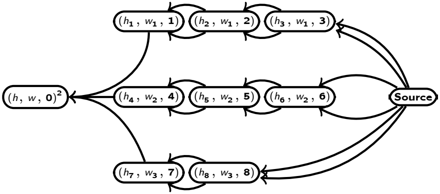


Reduce the size of the hypergraph

Other improvements







and many others...

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Numerical results

Experiments configuration

Two classes of instances are generated:

- $H = W = L = 500$.
- $h_i,\ w_i,\ l_i,\ p_i \in \mathcal{U}(80, 170)$.
- s_i and d_i are generated by cliques for the first class, $s_i,\ d_i \in \mathcal{U}(0, 1000)$ for the second class, similarly to [Caprara et al., 2013]

Group	$\lvert\mathcal{I}\rvert$
C10	~ 75
C15	~ 110
C30	~ 215
U50	50
U70	70
U100	100
U200	200

Table: Average number of items per group of instances

Machine setup: Haswell Intel Xeon E5-2680 v3 CPU at 2.5 GHz with 128 Go RAM.
Solver: CPLEX 20.1.
Time limit: 30 minutes.

Numerical results

Comparison of formulations

Group	NbVariables	NbConstraints	IP/LP Gap[1,2]	NbSolved
C10	17759	53230	5.97 %	1 / 10
C15	49068	151939	11.73 %	0 / 10
C30	363145	1120488	46.13 %	0 / 10
U50	5420	69781	0.14 %	9 / 10
U70	10438	181547	1.30 %	4 / 10
U100	21204	504690	11.41 %	1 / 10
U200	86361	3791192	70.08 %	1 / 10
Total	–	–	–	16 / 70

Table: Compact MIP formulation

Group	NbVertices	NbArcs	IP/LP Gap[1,2]	NbSolved
C10	146307	193277	4.48 %	2 / 10
C15	570665	698449	6.50 %	0 / 10
C30	5150654	5933438	31.33 %	0 / 10
U50	17183	32867	0.14 %	9 / 10
U70	30617	59912	1.30 %	3 / 10
U100	48738	102766	5.74 %	1 / 10
U200	150916	369086	3.27 %	4 / 10
Total	–	–	–	19 / 70

Table: Arc flow formulation

Both formulations have around the same number of constraints but the hypergraph formulation has around 13.3 times more variables.

[1] IP/LP Gap formula : $\frac{|\mathbf{LP} - \mathbf{BestInteger}|}{|\mathbf{BestInteger}|}$

[2] Only instances where the three formulations LPs could be solved are included

Numerical results

Impact of improvements on hypergraph formulation

Group	NbVertices	NbArcs	IP/LP Gap[1,2]	NbSolved
C10	146307	193277	4.48 %	2 / 10
C15	570665	698449	6.50 %	0 / 10
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U100	48738	102766	5.74 %	1 / 10
U200	150916	369086	3.27 %	4 / 10
Total	–	–	–	19 / 70

Table: Arc flow formulation

Group	NbVertices	NbArcs	IP/LP Gap[1,2]	NbSolved
C10	38398	55888	4.42 %	2 / 10
C15	156530	199988	6.39 %	0 / 10
C30	1860081	2157920	31.32 %	0 / 10
U50	2533	10250	0.14 %	10 / 10
U70	4467	17300	1.30 %	7 / 10
U100	5374	23790	5.27 %	7 / 10
U200	15976	96790	2.77 %	7 / 10
Total	–	–	–	33 / 70

Table: Improved arc flow formulation

The improved version has about 20% of the number of vertices and 29% of the number of arcs.

[1] IP/LP Gap formula : $\frac{|\mathbf{LP} - \mathbf{BestInteger}|}{|\mathbf{BestInteger}|}$

[2] Only instances where the three formulations LPs could be solved are included

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Conclusion

We have seen:

- Arc flow formulations on graphs and hypergraphs and the differences between both tools.
- A newly defined problem and a compact and an arc flow formulation for it.
- Several improvements useful to improve the linear relaxation and reduce the size of the arc flow formulation.

Perspectives:

- Propose further valid inequalities to improve the linear relaxation of the arc flow formulation.
- Thorough computational study on what reductions improve best the linear relaxation and the size of the formulation.

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